Coxeter Matroids

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What is a matroid?

Abstract definition (Basis)

Given a ground set $[n]$, a subset $\mathcal{B}(M) \subseteq \mathcal{P}([n])$ is the set of basis of a matroid M if satisfies the **exchange axiom**:

$$\forall A, B \in \mathcal{B}(M), a \in A - B \ \exists b \in B - A \ : \ A - \{a\} \cup b \in \mathcal{B}(M)$$

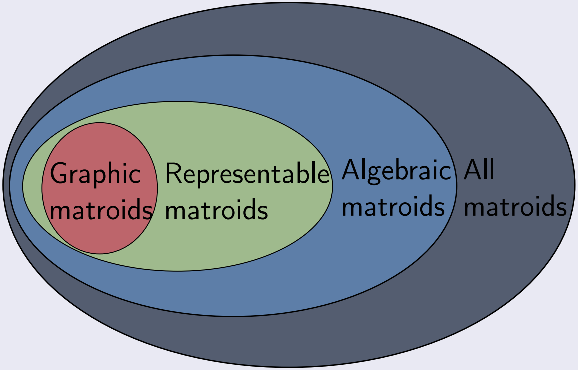
Rank

It can be seen that the cardinal of the elements of $\mathcal{B}(M)$ must be the same. We will call it $k = rank(M)$

The elements of $\mathcal{B}(M)$ are called *Maximal independent sets*. We will notice the intuition of this naming in later slides.

Matroids can be represented in several ways. The following diagram shows the different types of matroids and its relations.

Diagram of matroids

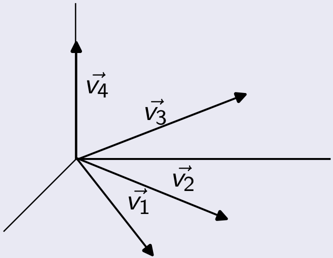


Representable Matroids

Definition

In a representable matroid we assign a vector to every element of the ground set $i \in [n] \to v_i \in V$. Now all the maximal independent sets of the matroids can be thought as sets of vectors that are a basis of a vector subspace $W \subseteq V$ of rank k.

Example



Independent sets

We notice that v_1, v_2, v_3 lie all in the same plane, so they are linearly dependent. We can make a basis of $\mathbb{R}^3$ in the following ways

$$\langle v_1, v_2, v_4\rangle, \langle v_1, v_3, v_4\rangle, \langle v_2, v_3, v_4\rangle$$

So the basis of the matroid are

$$124, 134, 234$$

Coxeter Groups

Definition

Given a set of generators $S = \{s_i\}$, a Coxeter group is a group whose presentation $\langle s_1, s_2, \ldots, s_n \rangle$ satisfy

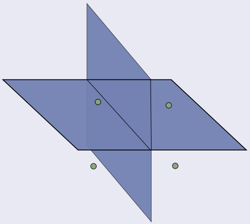
$$(s_i s_j)^{m_{ij}} = 1$$
$$m_{ii} = 1$$
$$m_{ij} \geq 2 \ \forall i \neq j$$

Hyperplanes

Coxeter groups are usually thought as finite groups of reflections into a vector space V. Each generator s_i is associated with a hyperplane ρ_i. Then, if we consider a vector in the vector space v, we can define the action of each element s_i of the group over v as the reflections by ρ_i.

$s_i v = v - 2\frac{\langle \rho_i, v \rangle}{\langle \rho_i, \rho_i \rangle} \rho_i$

Example



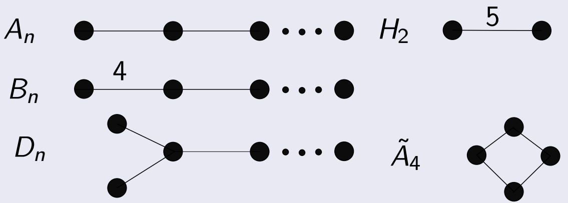
Dynkin diagrams

Dynkin diagram

We can assign every Coxeter group a diagram in the following way:

- Nodes are the set of generators S
- We connect nodes s_i, s_j by an edge if $m_{ij} \geq 3$
- We label the edges with m_{ij} if it is ≥ 4

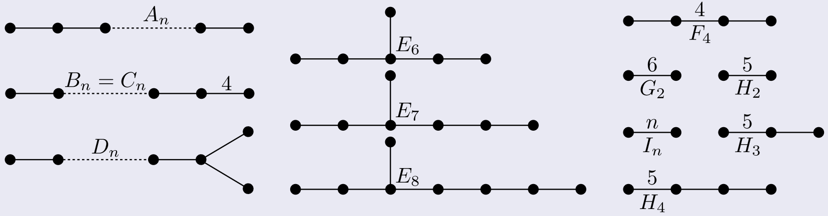
Examples of Dynkin diagrams



Coxeter Groups chart

All the finite Coxeter Groups can be classified in the following chart:

Chart



Definition of Coxeter Matorid

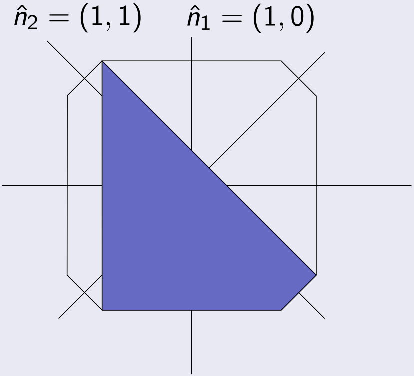
Let W a Coxeter group and F the set of vectors generated by the actions of W over a starting vector $\overline{x}$.

Let P the polytope generated by the convex hull of the set of vertices F.

Definition

A subset of vertices $U \subseteq F$ is a Coxeter Matroid if the convex hull Q generated by U has all the edges parallel to the edges of P.

Example of Coxeter Matroid



Rings

We should take into account that there is not a bijection between Dynkin diagrams and Polytopes generated by reflection. A key point is where do we place the first point we are considering the orbit.

Rings

Considering the base vector, we see if belongs some hyperplane:

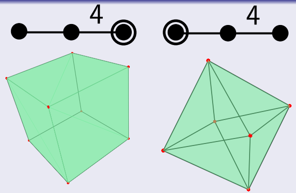
If $\overline{x} \in \rho_i$ keep the corresponding node untouched on Dynkin diagram

If $\overline{x} \notin \rho_i$ round the corresponding node with a ring on Dynkin diagram

Combinatorically equal polytopes

The resulting polytopes differ depending on the distribution of rings. Two polytopes with the same distribution of rings are **combinatorically** the same (although are not the same polytope).

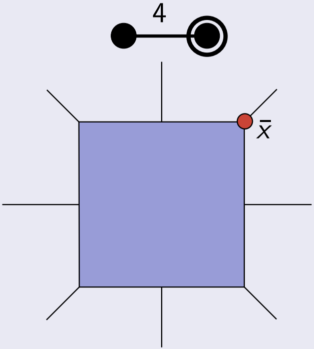
Example

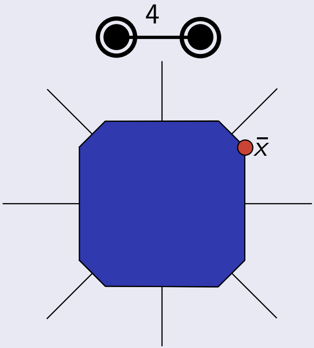


More examples of rings

We observe how the combinatoric class of the polytope changes if the vector $\overline{x}$ belongs to some hyperplane or not

Example



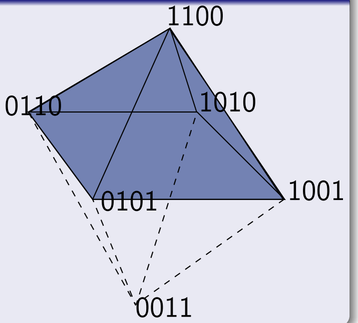


Type A_n

Matroid (base) polytope

A matroid (base) polytope is a geometrical representation of the basis of a matroid in a vector space.
Given a matroid M of rank k and the set of basis $\mathcal{B}(M) \subseteq \binom{[n]}{k}$, we assign each basis to its indicator vector $A \in \mathcal{B}(M) \to e_A = e_{i_1} + \ldots + e_{i_k} \in V$.
Thus, the convex hull of such vectors is a polytope $P \subseteq \Delta(n, k)$

Example



Matroid polytope

$$\mathcal{B}(M) = \{12, 13, 14, 23, 24\}$$

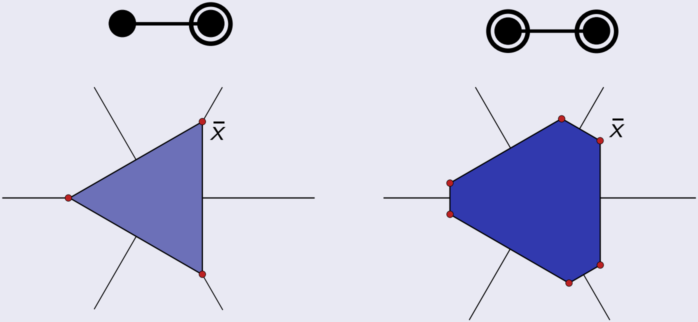
$$\text{Vertices} \begin{cases} 12 \to (1,1,0,0) \\ 13 \to (1,0,1,0) \\ 14 \to (1,0,0,1) \\ 23 \to (0,1,1,0) \\ 24 \to (0,1,0,1) \end{cases}$$

Types A_n

Withoff construction

The number of rings of the Dynking diagram establishes the numbers of ones of the vectors (k). Then, the corresponding polytope will differ depending on the distribution of the rings in Dynkin diagram.

Example



M-convex functions

Given a set of points T, we define a height function as $h : T \to \mathbb{R}$.

Definition (Regular subdivision)

A subset S of points is a (lower) regular subdivision induced by h if the convex hull of S is described by the lower convex hull of the polytope $T \times h(T)$

Definition (M-convex function)

The height function h is said to be *M-convex* if the regular subdivision induced by h is permutahedral (i.e. has the edges parallel to $e_i - e_j$).

We will use the notation $Sij := S \cup \{i\} \cup \{j\}$

3-Term Plücker Relations

Definition (3-Term Plücker Relations)

Let h be a height function on $\Delta(d,n)$. We say that 3TPR holds if for each $S \in \binom{[n]}{d-2}$ and $i,j,k,l \notin S$, the minimum

$$\min\left\{h(Sij)+h(Skl),\ h(Sik)+h(Sjl),\ h(Sil)+h(Sjl)\right\}$$

is attained at least twice.

Theorem

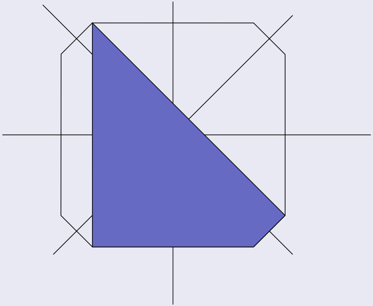
A height function induces a permutahedral regular division if the ***3-Term Plücker Relations*** *(3TPR) holds.*

Type B_n

The task is to find a condition similar to 3TPR for type B_n. That is, a condition for a height function to form a regular subdivision that partitions the polytope in subpolytopes whose edges are parallel to the original polytope.

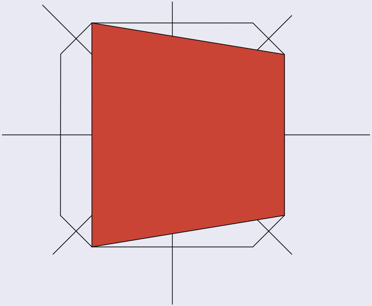
Coxeter Matroid

$\hat{n}_2 = (1,1) \quad \hat{n}_1 = (1,0)$



Not a Coxeter Matroid

$\hat{n}_2 = (1,1) \quad \hat{n}_1 = (1,0)$



Description of groups B_n

The group B_n is isomorphic to the group of signed permutations. Then the vertices can be thought as an state of a table with n labeled cards, each of one up or down.

Elements in S

We have two types of elements in S:

- τ corresponds to the change of sign of the first coordinate.
- s_i corresponds to the transposition and change of sign of the coordinates i and $i+1$

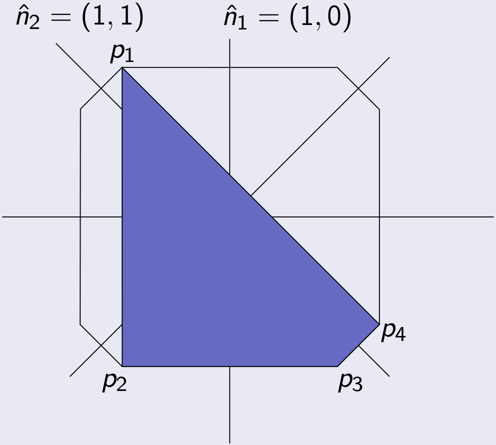
Theorem

A subpolytope $conv(U) = Q \subseteq P$ is a Coxeter Matroid if and only if all the edges of Q are or the form $p \to wp$ with w one of the following forms

- $w = c_i$, *where c_i means change the sign of i component*
- $w = t_{ij}$, *where t_{ij} is the transposition of the i, j components*
- $w = c_{ij} t_{ij}$. *Transposition and change of sign*

Example

Coxeter Matroid



Actions

$$p_1 \to p_2 = c_2 p_1$$
$$p_2 \to p_3 = c_1 p_2$$
$$p_3 \to p_4 = c_{12} t_{12} p_3$$
$$p_4 \to p_1 = t_{12} p_4$$

Bibliography

Michael Joswig, Georg Loho, Dante Luber, Jorge Alberto Olarte
Generalized Permutahedra and positive flag dressians
Arxiv 2111.13676v2

Alexandre Borovik, I.M. Gelfand, Neil White
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